

MATRIX CHARACTERIZATION OF MULTI-DIMENSIONAL SUBSHIFTS OF FINITE TYPE

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ABSTRACT. Let $X \subset A^{\mathbb{Z}^d}$ be a 2-dimensional subshift of finite type. We prove that any 2-dimensional multidimensional subshift of finite type can be characterized by a square matrix of infinite dimension. We extend our result to a general d -dimensional case. We prove that the multidimensional shift space is non-empty if and only if the matrix obtained is of positive dimension. In the process, we give an alternative view of the necessary and sufficient conditions obtained for the non-emptiness of the multidimensional shift space. We also give sufficient conditions for the shift space X to exhibit periodic points.

1. INTRODUCTION

The study of dynamical systems originated to facilitate the study of natural processes and phenomenon. Many natural phenomenon can be modeled as discrete dynamical systems and their long term behavior can be approximated using the modeled system. However, investigating a general discrete dynamical system is complex in nature and the long term behavior of the system cannot always be determined accurately. The uncertainty in predicting long term behavior introduces dynamical complexity in the system which in turn results in erroneous behavior of the modeled system. Thus, there is a need to develop tools to facilitate the study of a general dynamical system which are not erroneous and can model the physical system with the sufficient accuracy. Symbolic dynamics is one of such tools which are structurally simpler and can be used to model the physical system with desired accuracy. In one of the early studies, Jacques Hadamard used symbolic dynamics to study the geodesic flows on surfaces of negative curvature[4]. Claude Shannon applied symbolic dynamics to the field of communication to develop the mathematical theory of communication systems[6]. Since then the topic has

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found applications in areas like data storage, data transmission and planetary motion to name a few. The area has also found significant applications in various branches of science and engineering[2, 3]. Its simpler structure and easy computability can be used to investigate any general dynamical system. Infact, it is known that every discrete dynamical system can be embedded in a symbolic dynamical system with appropriate number of symbols [13]. Thus, to investigate a general discrete dynamical system, it is sufficient to study the shift spaces and its subsystems.

Multidimensional shift spaces has been a topic of interest to many researchers. In one of the early works, Schmidt investigated multidimensional subshifts of finite type over finite number of symbols. He proved that for a multidimensional subshift, it is algorithmically undecidable whether an allowed partial configuration can be extended to a point in the multidimensional shift space. He also observed that it is algorithmically undecidable to verify the non-emptiness of a multidimensional shift defined by a set of finite forbidden patterns. He also gave an example to show that a multidimensional shift space may not contain any periodic points[7]. These results unraveled the uncertainty associated with a multidimensional shift space and attracted attention of several researchers around the globe. As a result, several researchers have explored the field and a lot of work has been done[1, 5, 7, 8, 9, 10, 11, 12]. In [1], authors proved that multidimensional shifts of finite type with positive topological entropy cannot be minimal. Infact, if X is subshift of finite type with positive topological entropy, then X contains a subshift which is not of finite type, and hence contains infinitely many subshifts of finite type. Quas and Trow in the same paper proved that every shift space X contains an entropy minimal subshift Y , i.e., a subshift Y of X such that $h(Y) = h(X)$ [1]. While [5] investigated mixing properties of multidimensional shift of finite type, [8] investigated minimal forbidden patterns for multidimensional shift spaces. In [9], authors exhibit mixing \mathbb{Z}^d shifts of finite type and sofic shifts with large entropy. However, they establish that such systems exhibit poorly separated subsystems. They give examples to show that while there exists \mathbb{Z}^d mixing systems such that no non-trivial full shift is a factor for such systems, they provide examples of sofic systems where the only minimal subsystem is a single point. In [10], for multidimensional shifts with $d \geq 2$, authors proved that a real number $h \geq 0$ is the entropy of a \mathbb{Z}^d shift of finite type if and only if it is the infimum of a recursive

sequence of rational numbers. In [11], Hochman improved the result and showed that $h \geq 0$ is the entropy of a \mathbb{Z}^d effective dynamical system if and only if it is the lim inf of a recursive sequence of rational numbers. The problem of determining which class of shifts have a dense set of periodic points is still open. For two-dimensional shifts, Lightwood proved that strongly irreducible shifts of finite type have dense set of periodic points [12]. However, the problem is still open for shifts of dimension greater than two.

Let $A = \{a_i : i \in I\}$ be a finite set and let d be a positive integer. Let the set A be equipped with the discrete metric and let $A^{\mathbb{Z}^d}$, the collection of all functions $c : \mathbb{Z}^d \rightarrow A$ be equipped with the product topology. The function $\mathcal{D} : A^{\mathbb{Z}^d} \times A^{\mathbb{Z}^d} \rightarrow \mathbb{R}^+$ be defined as $\mathcal{D}(x, y) = \frac{1}{n+1}$, where n is the least natural number such that $x \neq y$ in $R_n = [-n, n]^d$, is a metric on $A^{\mathbb{Z}^d}$ and generates the product topology. For any $a \in \mathbb{Z}^d$, the map $\sigma_a : A^{\mathbb{Z}^d} \rightarrow A^{\mathbb{Z}^d}$ defined as $(\sigma_a(x))_k = x_{k+a}$ is a d -dimensional shift and is a homeomorphism. For any $a, b \in \mathbb{Z}^d$, $\sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$ and hence \mathbb{Z}^d acts on $A^{\mathbb{Z}^d}$ through commuting homeomorphisms. A set $X \subseteq A^{\mathbb{Z}^d}$ is σ_a -invariant if $\sigma_a(X) \subseteq X$. Any set $X \subseteq A^{\mathbb{Z}^d}$ is shift-invariant if it is invariant under σ_a for all $a \in \mathbb{Z}^d$. A non-empty, closed shift invariant subset of $A^{\mathbb{Z}^d}$ is called a shift space. If $Y \subseteq X$ is a closed, nonempty shift invariant subset of X , then Y is called a subshift of X . For any nonempty $S \subset \mathbb{Z}^d$, the projection map $\pi_S : A^{\mathbb{Z}^d} \rightarrow A^S$ defined as $\pi_S = A^{\mathbb{Z}^d}|_S$ projects the elements of $A^{\mathbb{Z}^d}$ to A^S . Any element in A^S is called a pattern over S . A pattern is said to be finite if it is defined over a finite subset of \mathbb{Z}^d . A pattern q over S is said to be extension of the pattern p over T if $T \subset S$ and $q|_T = p$. The extension q is said to be proper extension if $S \cap Bd(T) = \emptyset$, where $Bd(T)$ denotes the boundary of T . For a shift space X and any set $S \subset \mathbb{Z}^d$, the set $\mathcal{A}_S = \{x \in A^S : x = \pi_S(y), \text{ for some } y \in X\}$ is called the set of allowed patterns(in X) over S . The set $\mathcal{A} = \bigcup_{S \subset \mathbb{Z}^d} \mathcal{A}_S$ is called the set of allowed patterns or language for the shift space X . Given a set $S \subset \mathbb{Z}^d$ and a set of patterns \mathcal{P} in A^S , the set $X = X(S, \mathcal{P}) = \{x \in A^{\mathbb{Z}^d} : \pi_S \circ \sigma^n(x) \in \mathcal{P} \text{ for every } n \in \mathbb{Z}^d\}$ is a subshift generated by the patterns \mathcal{P} . If the set S is finite, the subshift generated is a subshift of finite type. Refer [1, 7, 8] for details.

Although the definition of a multidimensional shift is given in terms of the allowed patterns, an equivalent definition can be given

in terms of forbidden patterns. Such a definition provides an alternate view of the subshifts of finite type and in some cases can be more beneficial for further investigations. For the sake of completion, we provide the equivalent definition below.

For a shift space X and any set $S \subset \mathbb{Z}^d$, the set $\mathcal{F}_S = \{x \in A^S : x \neq \pi_S(y) \text{ for any } y \in X\}$ is called the set of forbidden patterns (in X) over S . The set $\mathcal{F} = \bigcup_{S \subset \mathbb{Z}^d} \mathcal{F}_S$ is called the set of forbidden patterns for the shift space X . For a given set of patterns \mathcal{F} (possibly over different subsets of \mathbb{Z}^d), define,

$$X = \{x \in A^{\mathbb{Z}^d} : \text{any pattern in } \mathcal{F} \text{ does not appear in } x\}$$

It can be seen that the set defined above is a shift space. If \mathcal{F} contains finitely many patterns defined over finite subsets of \mathbb{Z}^d , then the shift generated is a shift of finite type. We denote the shift space generated by the set of forbidden patterns \mathcal{F} by $X_{\mathcal{F}}$. Two forbidden sets \mathcal{F}_1 and \mathcal{F}_2 are said to be equivalent if they generate the same shift space, i.e. $X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$. A forbidden set \mathcal{F} of patterns is called minimal for the shift space X if \mathcal{F} is the set with least cardinality such that $X = X_{\mathcal{F}}$. It is worth mentioning that a shift space X is of finite type if its minimal forbidden set is a finite set of finite patterns.

Let M be a square matrix (possibly infinite) with indices $\{i : i \in \mathfrak{I}\}$. We say that the index i is u -related to j if $M_{ji} = 1$. Let the collection of indices u -related to j be denoted by R_j^u . We say that the indices j is d -related to i if $M_{ji} = 1$. Let the collection of indices d -related to i be denoted by R_i^d . It may be noted that i is u -related to j if and only if j is d -related to i . The set \mathfrak{I} is said to be complementary if for each $i \in \mathfrak{I}$, there exists $j, k \in \mathfrak{I}$ such that j is u -related to i and k is d -related to i .

In this paper we investigate some of the questions raised in [7]. In the process we address the problem of non-emptiness and existence of periodic points for a multidimensional shift of finite type. We prove that the any 2-dimensional shift of finite type can be characterized by an infinite square matrix of uncountable dimension. We extend our results to a general d -dimensional case. We provide necessary and sufficient conditions for a multidimensional subshift of finite type to be non-empty. In the end, we also give sufficient condition for the subshift to contain periodic points.

2. MAIN RESULTS

Proposition 1. *X is a d -dimensional shift of finite type \implies there exists a set C of d -dimensional cubes such that $X = X_C$.*

Proof. Let X be a shift of finite type and let \mathcal{F} be the minimal forbidden set of patterns for the shift space X . It may be noted that \mathcal{F} contains finitely many patterns defined over finite subsets of \mathbb{Z}^d . For any pattern p in \mathcal{F} , let l_p^i be the length of the pattern p in the i -th direction. Let $l_p = \max\{l_p^i : i = 1, 2, \dots, d\}$ denote the width of the pattern p and let $l = \max\{l_p : p \in \mathcal{F}\}$. Let \mathbb{C}_l be the collection of d -dimensional cubes of length l and let $\mathbb{E}_{\mathcal{F}}$ denote the set of extensions of patterns in \mathcal{F} . Let $C = \mathbb{C}_l \cap \mathbb{E}_{\mathcal{F}}$. It may be observed that if p is a pattern with width l , forbidding a pattern p for X is equivalent to forbidding all extensions q of p in \mathbb{C}_l . Thus, each pattern in the forbidden set of width l can be replaced by an equivalent forbidden set of cubes of length l and C is an equivalent forbidden set for the shift space X . Consequently, $X = X_C$ and the proof is complete. \square

Remark 1. The above result proves that every d -dimensional shift of finite type is generated by a set of cubes of fixed finite length. Such a consideration leads to an equivalent forbidden set which in general is not minimal. The above result constructs an equivalent forbidden set by considering all the cubes which are extension of the set of patterns in \mathcal{F} . However, the cardinality of the new set can be reduced by considering only those cubes which are not proper extensions of patterns in \mathcal{F} (but are of same size l). Such a construction reduces the cardinality of the forbidden set considerably and hence reduces the complexity of the system. It may be noted that the forbidden set obtained on reduction is still not minimal. However, the d -dimensional cubes generating the elements of X are of same size and can be used advantageously for constructing elements of X . We say that a shift of finite type X is generated by cubes of length l if there exists a set of cubes C of length l such that $X = X_C$.

Proposition 2. *Every 2-dimensional shift of finite type X can be characterized by an infinite square matrix.*

Proof. Let X be a 2-dimensional shift of finite type and let \mathcal{F} be the equivalent set of forbidden cubes (of fixed length, say l) for the space X . Let \mathcal{A} be the generating set of cubes (of length l) for the space X . It may be noted that as cubes of length l form a generating set for the shift space X , to verify whether any $x \in A^{\mathbb{Z}^2}$ belongs to X , it is

sufficient to examine strips of height l in x .

Let $\mathcal{A}^2 = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} : A, B \in \mathcal{A}, \begin{pmatrix} A \\ B \end{pmatrix} \text{ is allowed in } X \right\}$.

By construction, \mathcal{A}^2 is a finite set of $2n \times n$ allowed rectangles, say $\{a_1, a_2, \dots, a_k\}$, generating the shift space X .

Define a $k \times k$ matrix M as

$$M_{ij} = \begin{cases} 0, & (a_i a_j) \text{ is forbidden in } X; \\ 1 & (a_i a_j) \text{ is allowed in } X; \end{cases}$$

Then, the sequence space corresponding to the matrix M , $\Sigma_M = \{(x_n) : M_{x_i x_{i+1}} = 1, \forall i\}$ generates all allowed infinite strips (of height $2l$) in X . It may be noted that any element in Σ_M is element of the form $\begin{pmatrix} A \\ B \end{pmatrix}$, where A and B are allowed infinite strips of height l .

Generate an infinite matrix \mathfrak{M} , indexed by allowed infinite strips of height l , using the following algorithm:

- (1) Pick any $\begin{pmatrix} A \\ B \end{pmatrix} \in \Sigma_M$ and index first two rows and columns of the matrix by A and B . Set $m_{BA} = 1$.
- (2) For each $\begin{pmatrix} A \\ B \end{pmatrix} \in \Sigma_M$, if the rows and columns indexed A and B exist, set $m_{BA} = 1$. Else, label next row and/or column as A and/or B (whichever required) and set $m_{BA} = 1$.
- (3) In the infinite matrix generated in step 2, set $m_{BA} = 0$, if m_{BA} has so far not been assigned a value.
- (4) In the infinite matrix obtained, if there exists an index A such that the A -th row or column is zero, delete the A -th row and column from the matrix generated.

The above algorithm generates an infinite 0-1 matrix where $m_{BA} = 1$ if and only if $\begin{pmatrix} A \\ B \end{pmatrix}$ is allowed in X , where A and B are allowed infinite strips (of height l) in X . Let $\Sigma_{\mathfrak{M}}$ be the sequence space associated with the matrix \mathfrak{M} . Consequently, any sequence in $\Sigma_{\mathfrak{M}}$ gives a vertical arrangement of infinite allowed strips (of height l) such that the arrangement is allowed in X and hence generates an element in X . Conversely, any element in X is a sequential (vertical) arrangement of infinite strips of height l and hence is generated by a sequence in $\Sigma_{\mathfrak{M}}$. Consequently, $X = \Sigma_{\mathfrak{M}}$ and the proof is complete. \square

Remark 2. The above result characterizes elements of the shift space X by a infinite square matrix \mathfrak{M} . It may be noted that if row/column for an index A is zero, the algorithm deletes the row and column with index A . Such a criteria reduces the size of the matrix and will result in a matrix of dimension 0, if the shift space is empty. Further, the characterization of the space yields a matrix of infinite (uncountable) dimension. Consequently, it is undecidable whether a shift of finite type generated by set of cubes \mathcal{A} is non-empty. It may be noted that although the algorithm does not guarantee a positive dimensional matrix, if the shift space X is non-empty the matrix generated is definitely of positive dimension and characterizes the elements in X . Further, as each row/column of the matrix generated has atleast one non-zero entry, each block indexing the matrix can be extended to an element of X . Consequently, any submatrix of the matrix \mathfrak{M} cannot generate the shift space X . In light of the remark stated, we get the following result.

Corollary 1. *A 2-dimensional shift of finite type is non-empty if and only if the characterizing matrix \mathfrak{M} is of positive dimension. Further, any proper submatrix of the matrix \mathfrak{M} generates a proper subshift and hence the matrix \mathfrak{M} is minimal.*

Remark 3. Although, in general it is undecidable whether a multidimensional shift of finite type is non-empty, the non-emptiness problem can be addressed using submatrices of the matrix \mathfrak{M} . In particular, if there exists a submatrix \mathfrak{N} of \mathfrak{M} (say generated after finite/countable steps of algorithm) such that the space $\Sigma_{\mathfrak{N}}$ is non-empty, the shift space X is non-empty. As any non-empty shift space characterized by a finite dimensional matrix contains periodic points, such a verification (in finite time) cannot be conducted for a shift space without periodic points. Further, such a verification addresses only the non-emptiness problem and does not characterize the elements in the shift space.

Remark 4. For a shift space X , with generating set of cubes of height l , let \mathcal{L} denote set of all allowed infinite strips of height l . Recalling the notions of u -related indices for a square matrix M , for any two infinite strips A, B of height l , we say that A is u -related (d -related) to B if A and B are indices of M such that $M_{BA} = 1$ ($M_{AB} = 1$). Further, generalizing the definition, a family of allowed infinite strips of height l is complementary if for each A in \mathcal{L} there exists infinite strips $B, C \in \mathcal{L}$ such that B is u -related to A and C is d -related to A . Thus, the algorithm generates u -related (d -related) infinite strips for the shift space X which in turn generates an arbitrary element of X .

As any element of the shift space is a sequential arrangement of u -related (d -related) infinite strips, the characterization of the elements of the space X by a matrix \mathfrak{M} is equivalent to finding all the u -related (d -related) pairs of infinite strips for the space X . As any infinite strip of height l (say A) can be extended to an element of X only if there exists infinite strips B, C of height l such that B is u -related to A and C is d -related to A , only members of complementary family can form the building blocks for an element of X . As a result, we get the following corollary.

Corollary 2. *Let X be a multidimensional shift space generated by cubes of length l and let \mathfrak{B} be the infinite strips of height l allowed in X . Then, the shift space X is non-empty if and only if there exists non-empty set of indices $\mathfrak{B}_0 \subseteq \mathfrak{B}$ such that \mathfrak{B}_0 is complementary.*

Remark 5. The above result provides an alternate view of the criteria established for the non-emptiness of the space X . The result does not require the matrix \mathfrak{M} for establishing the non-emptiness for the shift space. The set of indices during construction of the matrix may be observed at each iteration and existence of a complementary subfamily can be used to establish the non-emptiness of the space X . However, as the algorithm does not provide any optimal technique for picking the block $\begin{pmatrix} A \\ B \end{pmatrix}$ at each iteration, such a consideration does not reduce the time complexity and the problem of non-emptiness is still undecidable. However, algorithms for optimal selection of the infinite blocks $\begin{pmatrix} A \\ B \end{pmatrix}$ may be proposed which in turn may reduce the time complexity of the algorithm. As any multidimensional shift can be realized as an extension of a 2-dimensional shift, similar results are true for a general d -dimensional shift of finite type. For the sake of completion, we include the proof of main result below.

Proposition 3. *If X is a d -dimensional shift of finite type, then the elements of X can be determined by an infinite square matrix.*

Proof. Let X be a d -dimensional shift of finite type and let \mathcal{F} be the equivalent set of forbidden cubes (of fixed length, say l) for the space X . Let \mathcal{A} be the generating set of cuboids of size $\underbrace{2l \times 2l \times \dots \times 2l}_{d-1 \text{ times}} \times l$ for

the space X .

By construction, \mathcal{A} is a finite set of allowed rectangles, say $\{a_1, a_2, \dots, a_k\}$. Define a $k \times k$ matrix \mathfrak{M}^0 as

$$\mathfrak{M}_{ij}^0 = \begin{cases} 0, & (a_i a_j) \text{ is forbidden in } X; \\ 1 & (a_i a_j) \text{ is allowed in } X; \end{cases}$$

where $(a_i a_j)$ denotes adjacent placement of a_j with a_i in the positive d -th direction.

Then, the sequence space corresponding to the matrix \mathfrak{M}^0 , $\Sigma_{\mathfrak{M}^0} = \{(x_n) : \mathfrak{M}_{x_i x_{i+1}}^0 = 1, \forall i\}$ generates all allowed one directional (in d -th direction) infinite strips in X .

It may be noted that any element in $\Sigma_{\mathfrak{M}^0}$ is element of the form $\begin{pmatrix} A \\ B \end{pmatrix}_0$, where A and B are allowed infinite strips (in direction d) of di-

mension $\underbrace{2l \times 2l \times \dots 2l}_{d-2 \text{ times}} \times l \times \infty$ and $\begin{pmatrix} A \\ B \end{pmatrix}_0$ denotes adjacent placement of B with A in the negative $d - 1$ -th direction.

Generate an infinite matrix \mathfrak{M}^1 , indexed by allowed infinite strips of dimension $\underbrace{2l \times 2l \times \dots 2l}_{d-2 \text{ times}} \times l \times \infty$, using the following algorithm:

- (1) Pick any $\begin{pmatrix} A \\ B \end{pmatrix}_0 \in \Sigma_{\mathfrak{M}^0}$ and index first two rows and columns of the matrix by A and B . Set $m_{BA} = 1$.
- (2) For each $\begin{pmatrix} A \\ B \end{pmatrix}_0 \in \Sigma_{\mathfrak{M}^0}$, if the rows and columns indexed A and B exist, set $m_{BA} = 1$. Else, label next row and/or column as A and/or B (whichever required) and set $m_{BA} = 1$.
- (3) In the infinite matrix generated in step 2, set $m_{BA} = 0$, if m_{BA} has so far not been assigned a value.
- (4) In the infinite matrix obtained, if there exists an index A such that the A -th row or column is zero, delete the A -th row and column from the matrix.

The above algorithm generates an infinite 0-1 matrix where $m_{BA} = 1$ if and only if $\begin{pmatrix} A \\ B \end{pmatrix}_0$ is allowed in X , where A and B are of dimension $\underbrace{2l \times 2l \times \dots 2l}_{d-2 \text{ times}} \times l \times \infty$. Let $\Sigma_{\mathfrak{M}^1}$ denote the sequence space corresponding to the matrix generated above. It can be seen that the space $\Sigma_{\mathfrak{M}^1}$ precisely is the collection of allowed bi-infinite strips (in direction d and $d - 1$). Further, as any element in $\Sigma_{\mathfrak{M}^1}$ is of the form $\begin{pmatrix} A \\ B \end{pmatrix}_1$, where

A and B are allowed infinite strips (in direction d and $d - 1$) of dimension $\underbrace{2l \times 2l \times \dots 2l}_{d-3 \text{ times}} \times l \times \infty \times \infty$ and $\begin{pmatrix} A \\ B \end{pmatrix}_1$ denotes adjacent placement of B with A in the negative $d - 2$ -th direction, a repeated application of the algorithm generates a matrix \mathfrak{M}^2 which extends the infinite patterns in $\Sigma_{\mathfrak{M}^1}$ along the direction $d - 3$ to generate the space $\Sigma_{\mathfrak{M}^2}$. Consequently, repeated application of the above algorithm extends the allowed patterns infinitely in all the d directions (one direction at each step) to obtain a point in X . Further, as any point in X can be visualized as such an extension of allowed cubes in the d directions, the matrix obtained (at the final step) characterizes the elements of the space X .

□

Remark 6. The above result characterizes the multidimensional shift space by a matrix \mathfrak{M} . The result is a repeated application of the 2-dimensional case, extending the allowed block in each of the d directions. In the process, at each step i we obtain an infinite matrix characterizing the extension of an allowed block in the i -th direction. Although the rows and columns of the characterizing matrix \mathfrak{M} are indexed by infinite blocks allowed in X , their existence/verification is beyond any ambiguity as they are algorithmically generated. It may be noted that extension in any of the directions (at step i) does not guarantee an extension to the element of X . In particular, a block extendable in a direction i (or in a few directions i_1, i_2, \dots, i_r) need not necessarily extend to an element in X . In particular if the shift space is empty, although we may obtain matrices of positive dimension in initial few steps, the final matrix obtained characterizing the elements of X is 0-dimensional. Consequently, once again, the shift space is non-empty if and only if the matrix generated (at the final step) is of positive dimension. As the algorithm is an extension of the algorithm for the 2-dimensional case, results similar to the 2-dimensional case also hold good for any general dimension $d \geq 3$. For the sake of completion, we mention the generalizations below.

Corollary 3. *A multidimensional shift of finite type is non-empty if and only if the characterizing matrix \mathfrak{M} is of positive dimension. Further, any proper submatrix of the matrix \mathfrak{M} generates a proper subshift and hence the matrix \mathfrak{M} is minimal.*

Corollary 4. *Let X be a multidimensional shift space and let \mathfrak{B} be the infinite strips of height l allowed in X . Then, the shift space X is non-empty if and only if there exists $\mathfrak{B}_0 \subseteq \mathfrak{B}$ such that \mathfrak{B}_0 is complementary.*

We now give some results relating the matrix M and the dynamical behavior of the shift space X .

Proposition 4. *Let X be a multidimensional shift space and let \mathcal{B} be the infinite strips of height l allowed in X . If there exists a finite complementary set $\mathcal{B}_0 \subset \mathcal{B}$, then the set of periodic points is non-empty.*

Proof. Let \mathcal{B} be the infinite strips of height l allowed in X and let $\mathcal{B}_0 \subset \mathcal{B}$ be a finite complementary set. By definition, elements of \mathcal{B}_0 form indices (not all) for the matrix M . Let \mathfrak{M} be the submatrix of M indexed by elements of \mathcal{B}_0 . As the set \mathcal{B}_0 is complementary, the shift generated by \mathcal{B}_0 (say $\Sigma_{\mathcal{B}_0}$) is non-empty. Further, as shift defined by a finite dimensional matrix contains periodic points, there exists periodic points for $\Sigma_{\mathcal{B}_0}$ (and hence for the shift space X). \square

3. CONCLUSION

In this paper, we investigate the non-emptiness problem for a multidimensional shift space of finite type. In the process, we prove that any multidimensional shift of finite type can be characterized by an infinite square matrix of uncountable dimension. We prove that the multidimensional shift space is non-empty if and only if the matrix M is of positive dimension. We also prove that any submatrix of the matrix obtained generates a proper subshift of X and hence the matrix M minimally generates the elements of X . We further observe that non-emptiness of such a shift may be examined using complementary set of indices. However, construction of such a family of indices is non-trivial and may not be possible in finite time. Consequently, the non-emptiness problem for such a space is undecidable. We also provide a sufficient condition for a multidimensional shift of finite type to exhibit periodic points.

REFERENCES

- [1] **A. Quas, P. Trow**, Subshifts of Multidimensional shifts of finite type, *Ergodic Theory and Dynamical Systems*, 20 (03), 2000, 859 - 874.
- [2] **Bruce P.Kitchens**, Symbolic Dynamics: One-Sided, Two-Sided and Countable State Markov Shifts, *Universitext. Springer-Verlag*, Berlin, 1998.
- [3] **D. Lind, B. Marcus** An introduction to symbolic dynamics and coding, *Cambridge University Press*, Cambridge, 1995.
- [4] **J. Hadamard**, Les surfaces a courbures opposes et leurs lignes godesiques, *J. Math. Pures Appl.*, 5) IV(1898), 27-74.
- [5] **J.C.Ban, W.G.Hu, S.S.Lin and Y.H.Lin** Verification of mixing properties in two-dimensional shifts of finite type, preprint arXiv:1112.2471.

- [6] **C.E. Shannon** A mathematical theory of communication, *Bell Syst. Tech. J.* 27 (1948), 379423, 623656.
- [7] **K. Schmidt**, Multi-dimensional symbolic dynamical systems, *Codes, Systems and graphical models*, (Minneapolis, MN, 1999), Springer, New York, 2001, pp. 67-82.
- [8] **Marie-Pierre Beal, Francesca Fiorenzi, Filippo Mignosi**, Minimal Forbidden Patterns of Multi-dimensional Shifts, *Int. J. Algebra Comput.*, 15, 73 (2005), DOI: 10.1142/S0218196705002165.
- [9] **Mike Boyle, Ronnie Pavlov, and Michael Cchraudner**, Multidimensional Sofic Shifts without Separation and their Factors, *Transactions of the American Mathematical Society*, Volume 362, Number 9, September 2010, Pages 46174653.
- [10] **Michael Hochman, Tom Meyerovitch**, A Characterization of the Entropies of Multidimensional Shifts of Fnite type, *Annals of Mathematics*, Vol. 171 (2010), No. 3, 2011-2038.
- [11] **Michael Hochman**, On Dynamics and Recursive Properties of Multidimensional Symbolic Dynamics, *Inventiones Mathematica*, Volume 176, Number 1, April, 2009.
- [12] **S.Lightwood**, Morphisms from non-periodic Z^2 -subshifts I: Constructing embeddings from homomorphisms, *Ergodic Theory Dynam. Systems*, 23 (2003), no. 2, 587-609.
- [13] **Xin-Chu Fu, WeiPing Lu, Peter Ashwin, Jinqiao Duan**, Symbolic Representations of Iterated Maps, *Topological Methods in Nonlinear Analysis*, Journal of the Juliusz Schauder Center Volume 18, 2001, 119-147.

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